CS331, Fall 2025
Lecture 4 (9/8)

Today:
- Word RAM
- Largest jump
- Largest subsequence sum
- Balanced parentheses

Fibonacci (Part III, Section 1)

(From last time...)

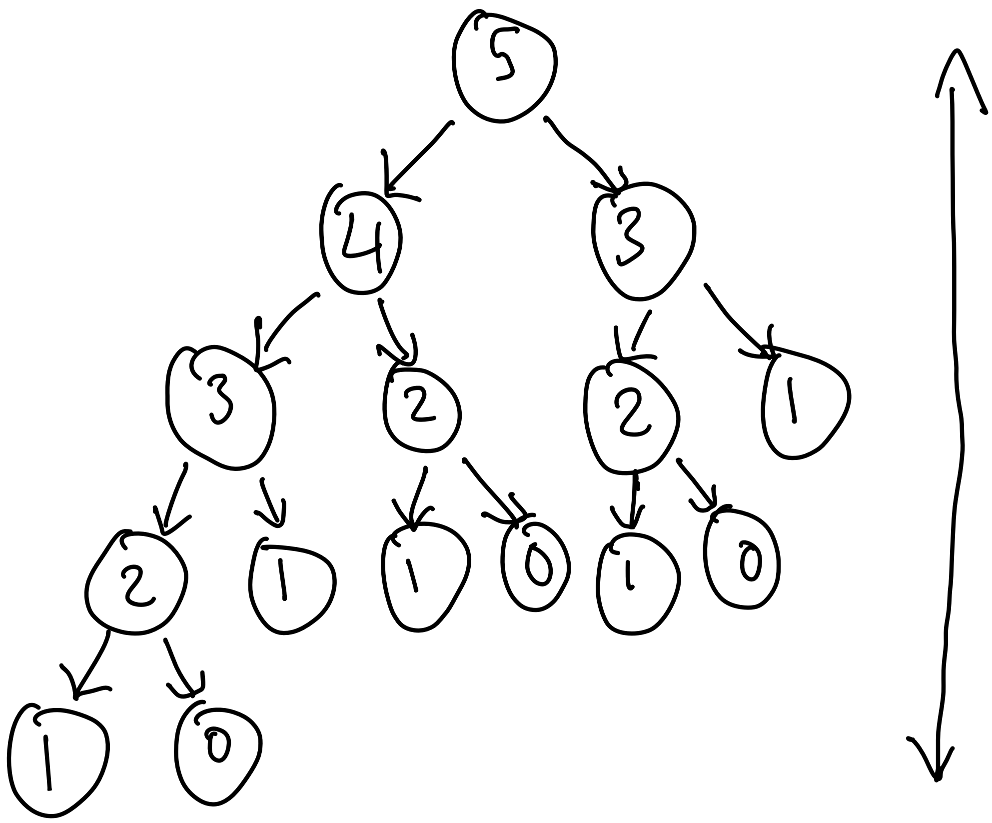
Input: $n \in \mathbb{N}$ Output: F_n, n^{th} Fibonacci number

First try:

```
FibNaive(n):
    If n == 0: Return 1
    If n == 1: Return 2
    Return FibNaive(n-1) + FibNaive(n-2)
```

✗ Warning: this will take exponential time!

e.g. FibNaive(5):



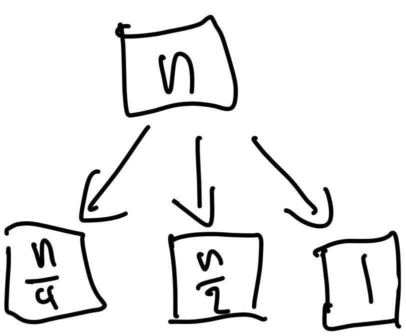
depth = n

branching factor ≈ 2

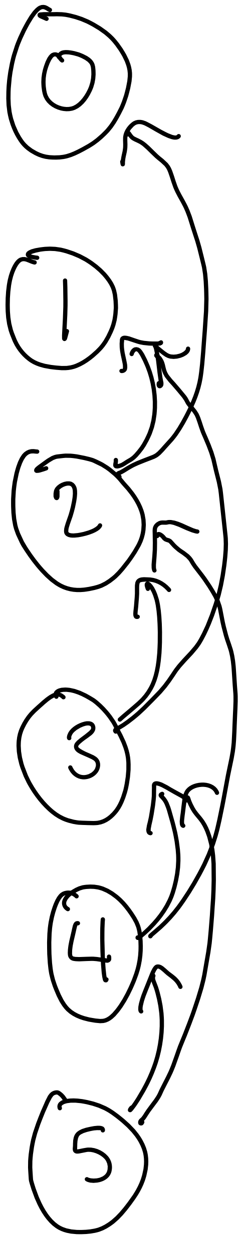
Idea: do things bottom-up to avoid recomputation

Intuition: induction is typically proven from bottom-up.

In recursive algos ("strong induction") multiple larger calls can use the same smaller call.



if every algo uses base case, save it & reuse it!



Dependency graph for Fib

"Memoized" implementation

```
Fib(n):
    L ← Array.Init(n+1)        // L[i] = F_{i+1}
    L[1] ← 1
    L[2] ← 2
    For 3 ≤ i ≤ n+1: L[i] ← L[i-1] + L[i-2]
    Return L[n+1]
```

Key takeaways:

1) order matters!
   (top-down vs. bottom-up)
2) save your work

L:

1	2	3	5	8	...

"memos"

Dynamic programming:

$\boxed{\text{Smarter and more flexible recursion}}$

(up to now, focus is "divide-and-conquer")

Applications:

- Coding interviews (seriously.)
- Reinforcement learning (founder: Bellman)
- Game theory / economics

Basic idea: 1) define subproblems you want to save / "memoize"

2) define order to solve them

It can be hard to understand at first.

Payoffs enormous: exponential $\Rightarrow$ near-linear time?

Next 4 lectures: many examples as a field guide

Word RAM model (Part I, Section 7)

Warmup: digits

How many digits is $n \in \mathbb{N}$?

Base 10: $1 + \lfloor \log_{10}(n) \rfloor = O(\log_{10}(n))$

Base 2: $1 + \lfloor \log_{2}(n) \rfloor = O(\log_{2}(n))$

Base b: $1 + \lfloor \log_{b}(n) \rfloor = O(\log_{b}(n))$

constant

all the same!

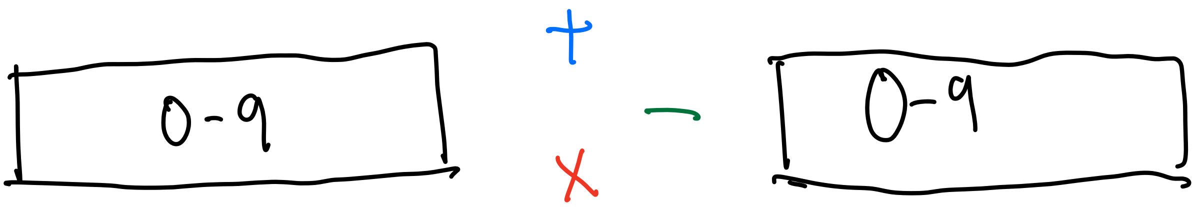
Recall that $\log_a(b) \log_b(c) = \log_a(c)$

Proof: $\left(a^{\log_a(b)}\right)^{\log_b(c)} = b^{\log_b(c)} = c = a^{\log_a(c)}$

Hence, $\dfrac{\log_2(n)}{\log_b(n)} = \underbrace{\log_2(b)}_{\text{constant}}$

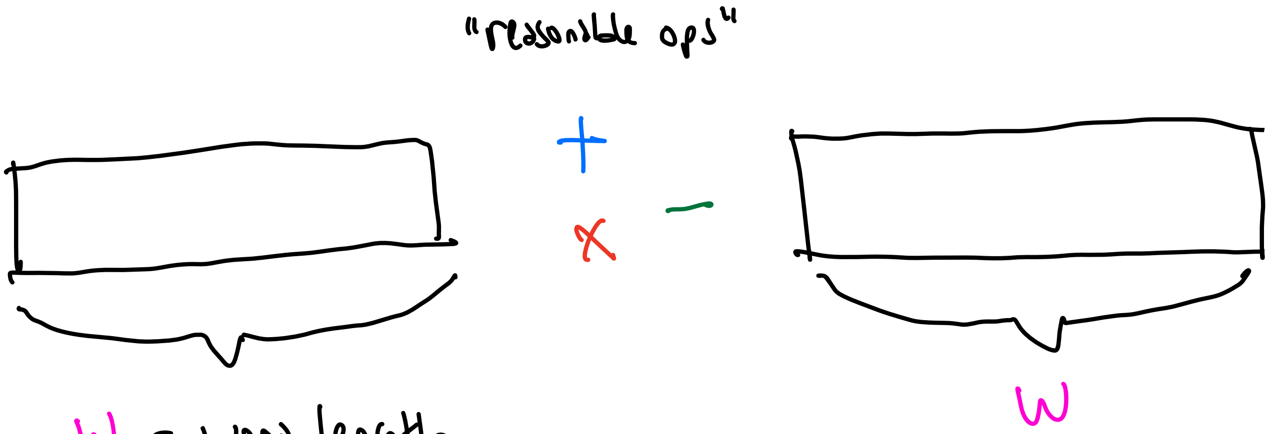
We'll be lazy & just write "$\log(n)$"

Bit complexity model: bit/digit ops take $O(1)$ time.



So, $\underbrace{\text{Fib}(n)}_{O(n)\text{ digits}} + \underbrace{\text{Fib}(n)}_{O(n)\text{ digits}}$ takes $O(n)$ time.

Word RAM model:



W = word length

Assume: takes $O(1)$ time.

Think: $W = 64$, maybe few 100's tops.

Ok to assume $\log(n) \leq W$.

Will do henceforth unless stated otherwise

Unreasonable to assume $n \leq w$. Keep an eye out!

Examples

- Incrementing a for loop counter $i \in (n)$ is $\log(n)$ digits, $O(1)$ time.
- Comparing two numbers in sorting if both $\underbrace{\leq 2^w}_{\text{by default, OK to assume}}$, $O(1)$ time.

Largest jump (Part III, Section 2.1)

Input: L is a list of n elements in $\mathbb{R}$

Output: (i, j) with $1 \leq i \leq j \leq n$

maximizing $L[j] - L[i]$

1	2	3	4	5	6	7
8	12	4	9	17	1	13

Example from last class

Buy ↓ / Sell →	1	2	3	4	5	6	7
1	$\boxed{0}$	$\boxed{4}$	-4	1	9	-7	5
2		0	-8	-3	5	-11	1
3			$\boxed{0}$	$\boxed{5}$	$\boxed{13}$	-3	9
4				0	8	-8	4
5					0	-16	-4
6						$\boxed{0}$	$\boxed{12}$
7							0
Best:	0	4	0	5	13	0	12

Time: $O(n^2)$ (for each i, j, compute $L[j]-L[i]$)

How to improve?

Idea: What do we need to know for $Best[j]$?

$$Best[j] = \max_{i\in[j]} L[j]-L[i] = L[j] - \underbrace{\min_{i\in[j]} L[j]}_{\text{maintain}}$$

Faster algo, take 1:

① Pass over list, maintain running min.

e.g.							
	8	12	4	9	17	1	13
MinUpTo	8	8	4	4	4	1	1

$$\text{MinUpTo}[j] = \min_{i \in [j]} L(i)$$

② Compute all $\text{Best}[j] = L[j] - \text{MinUpTo}[j]$

Both steps $O(n)$ time. Why?

$$\text{MinUpTo}[j] = \min\Big(\underbrace{\text{MinUpTo}[j-1]}_{\text{memoized}},\ L[j]\Big)$$

Faster algo, take 2:

All subproblems: (i, j), $n + \binom{n}{2} = \Theta(n^2)$ of them.

Special subproblems: (i^*, j) where $i^* = \text{MinUpTo}[j]$

$$\text{Best}(j) = L[j] - L(i^*).$$

only $O(n)$ of them.

Are special subproblems harder?

Not with memoization!

$$\text{Best}(j) = \max\left(0,\ \text{Best}(j-1) + \underbrace{L[j] - L[j-1]}_{\text{difference in sell price}}\right)$$

buy on day j
$i^* = j$

buy on day $< j$, keep
i^* the same

No need for MinUpTo. $O(1)$ time per $\text{Best}(j)$
$= O(n)$ total.

General strategy:

1) Design any correct algo.
2) Repeated structure? Memoize!
3) Go back to step 1). Simplify.

(the hardest step)

Two common forms
- Prefixes
- Multidimensional

Largest subsequence sum (Part III, Section 2.2)

Input: L is a list of n elements in $\mathbb{R}$

Output: (i, j) with $1 \le i \le j \le n$

maximizing $\underbrace{L[i] + L[i+1] + \dots + L[j-1] + L[j]}_{\text{subsequence sum}}$

Example

25	-60	7	-13	30

First try:

Start ↓ \ End →	1	2	3	4	5
1	$\boxed{25}$	$\boxed{-35}$	-28	-41	-11
2		-60	-53	-66	-36
3			$\boxed{7}$	$\boxed{-6}$	24
4				-13	17
5					$\boxed{30}$

Naive: $\underbrace{O(n^2)}_{\text{\# subproblems}} \times \underbrace{O(n)}_{\text{time / subproblem}} = O(n^3)$

Better: $O(n^2) \times \underbrace{O(1)}_{\text{proceed row by row, left-to-right.}} = O(n^2)$

$$\sum_{k=i}^{j} L(k) = \underbrace{\sum_{k=i}^{j-1} L[k]}_{\text{memoized}} + L(j)$$

Time to simplify.

Can we define $O(n)$ special subproblems?

Best(j) = largest subsequence sum ending on j

Recursive formula:

$$\text{Best}(j) = \max\left(L(j),\ \text{Best}(j-1) + L(j)\right)$$

don't include $L(j-1)$

include $L(j-1)$. may as well continue in best possible way

Again, $O(1)$ per subproblem using memoization

$\Rightarrow$ LSS in $O(n)$ time. :)

(Kadane, at a CMU seminar)

Balanced parentheses (Part III, Section 2.3)

Input: L is sequence of n chars: (or)

assume even

Output: True or False,

Can we match parentheses s.t.

each (paired with a later) ?

Examples

( ) (( ))) (

(((( ))))

(( ) ( ) (( )

Naive strategy: try all possible matchings

$$\binom{n}{2}\binom{n-2}{2}\cdots\binom{4}{2}\binom{2}{2} = \frac{n!}{2^n} \gg \text{poly}(n)$$

How to solve in polynomial time? Multidimensional DP!

$S[i][j]$ = True or False,

Can we balance $\underbrace{L[i:j]}_{\text{subarray between } L[i], L[j]}$

Recursive definition: Who is $L[i]$ paired to?

$\underset{i}{(}$ balanced $\underset{j}{)}$ OR $\underset{i}{(}$ balanced $\underset{k}{)}$ $\underset{k+1}{(}$ balanced $\underset{j}{)}$

$S[i][j]$ = True iff one of following holds

- $L[i] = ($ AND $L[j] =)$ AND $S[i+1][j-1]$
- $S[i][k]$ AND $S[k+1][j]$

Recursion order: by length. $O(n^2) \times \underbrace{O(n)}_{\text{possible pairings of } i} = O(n^3)$